In this case, P_i is the number hired in period i.

The following example provides a simplified illustration of a single-product, multi-period planning situation.

9.2 A Dynamic Production Problem

A company produces one product for which the demand for the next four quarters is predicted to be:

Spring	Summer	Autumn	Winter
20	30	50	60

Assuming all the demand is to be met, there are two extreme policies that might be followed:

1. "Track" demand with production and carry no inventory.
2. Produce at a constant rate of 40 units per quarter and allow inventory to absorb the fluctuations in demand.

There are costs associated with carrying inventory and costs associated with varying the production level, so one would expect the least-cost policy is probably a combination of (1) and (2) (i.e., carry some inventory, but also vary the production level somewhat).

For costing purposes, the company estimates changing the production level from one period to the next costs \$600 per unit. These costs are often called "hiring and firing" costs. It is estimated that charging \$700 for each unit of inventory at the end of the period can accurately approximate inventory costs. The initial inventory is zero and the initial production level is 40 units per quarter. We require these same levels be achieved or returned to at the end of the winter quarter.

We can now calculate the production change costs associated with the no-inventory policy as:

$$\$600 \times (20 + 10 + 20 + 10 + 20) = \$48{,}000.$$

On the other hand, the inventory costs associated with the constant production policy is:

$$\$700 \times (20 + 30 + 20 + 0) = \$49{,}000.$$

The least cost policy is probably a mix of these two pure policies. We can find the least-cost policy by formulating a linear program.

9.2.1 Formulation

The following definitions of variables will be useful:

P_i = number of units produced in period i, for i = 1, 2, 3, and 4;
I_i = units in inventory at the end of period i;
U_i = increase in production level between period $i - 1$ and i;
D_i = decrease in production level between $i - 1$ and i.

The P_i variables are the obvious decision variables. It is useful to define the I_i, U_i, and D_i variables, so we can conveniently compute the costs each period.

To minimize the cost per year, we want to minimize the sum of inventory costs:

$$\$700\, I_1 + \$700\, I_2 + \$700\, I_3 + \$700\, I_4$$

plus production change costs:

$$\$600\, U_1 + \$600\, U_2 + \$600\, U_3 + \$600\, U_4 + \$600\, U_5$$
$$+ \$600\, D_1 + \$600\, D_2 + \$600\, D_3 + \$600\, D_4 + \$600\, D_5.$$

We have added a U_5 and a D_5 in order to charge for the production level change back to 40, if needed at the end of the 4th period.

9.2.2 Constraints

Every multi-period problem will have a "material balance" or "sources = uses" constraint for each product per period. The usual form of these constraints in words is:

beginning inventory + production − ending inventory = demand.

Algebraically, these constraints for the problem at hand are:

$$P_1 - I_1 = 20$$
$$I_1 + P_2 - I_2 = 30$$
$$I_2 + P_3 - I_3 = 50$$
$$I_3 + P_4 = 60$$

Notice I_4 and I_0 do not appear in the first and last constraints, because initial and ending inventories are required to be zero.

If the formulation is solved as is, there is nothing to force U_1, D_1, etc., to be greater than zero. Therefore, the solution will be the pure production policy. Namely, $P_1 = 20$, $P_2 = 30$, $P_3 = 50$, $P_4 = 60$. This policy implies a production increase at the end of every period, except the last. This suggests a way of forcing U_1, U_2, U_3, and U_4 to take the proper values is to append the constraints:

$$U_1 \geq P_1 - 40$$
$$U_2 \geq P_2 - P_1$$
$$U_3 \geq P_3 - P_2$$
$$U_4 \geq P_4 - P_3.$$

Production decreases are still not properly measured. An analogous set of four constraints should take care of this problem, specifically:

$$D_1 \geq 40 - P_1$$
$$D_2 \geq P_1 - P_2$$
$$D_3 \geq P_2 - P_3$$
$$D_4 \geq P_3 - P_4.$$

To incorporate the requirement that the production level be returned to 40 at the end of the winter quarter, we add the variables U_5 and D_5 to measure changes at the end of the last quarter. U_5 and D_5 are forced to take on the right values with the constraints:

$$U_5 \geq 40 - P_4$$
$$D_5 \geq P_4 - 40.$$

Before moving on, we will note the production-change constraints can be reduced to 5 constraints from the 10 implied by the above form. The key observation is two constraints such as:

$$U_2 \geq P_2 - P_1$$
$$D_2 \geq P_1 - P_2$$

can be replaced by the single constraint:

$$U_2 - D_2 = P_2 - P_1.$$

The argument is more economic than algebraic. The purpose with either formulation is to force $U_2 = P_2 - P_1$ if $P_2 - P_1 \geq 0$ and $D_2 = P_1 - P_2$ if $P_1 - P_2 \geq 0$. From economics, you can argue that, at the optimal solution, you will find at most one of U_2 and D_2 are greater than 0 under either formulation. If both U_2 and D_2 are greater than 0 under the second formulation, then both can be reduced by an equal amount. Thus, reducing costs without violating any constraints.

The complete formulation is:

```
MODEL:
  !Minimize inventory + workforce change costs;
  MIN = 700 * I1 + 700 * I2 + 700 * I3 + 700 * I4
      + 600 * U1 + 600 * U2 + 600 * U3 + 600 * U4
      + 600 * D1 + 600 * D2 + 600 * D3 + 600 * D4
      + 600 * U5 + 600 * D5;
  !Initial conditions on inventory & production;
  [CNDBI] I0 = 0;
  [CNDBP] P0 = 40;
  !Beginning inventory + production = demand + ending inventory;
  [INV1] I0 + P1 = 20 + I1;
  [INV2] I1 + P2 = 30 + I2;
  [INV3] I2 + P3 = 50 + I3;
  [INV4] I3 + P4 = 60 + I4;
  !Change up - change down = prod. this period - prod. prev. period;
  [CHG1] U1 - D1 = P1 - P0;
  [CHG2] U2 - D2 = P2 - P1;
  [CHG3] U3 - D3 = P3 - P2;
  [CHG4] U4 - D4 = P4 - P3;
  [CHG5] U5 - D5 = P5 - P4;
  !Ending conditions;
  [CNDEI] I4 = 0;
  [CNDEP] P5 = 40;
END
```

The solution is:

```
Optimal solution found at step:         7
Objective value:                 43000.00
Variable           Value        Reduced Cost
      I1        5.000000           0.0000000
      I2       0.0000000            200.0000
      I3        5.000000           0.0000000
      I4       0.0000000           0.0000000
      U1       0.0000000            1200.000
      U2       0.0000000            250.0000
      U3        30.00000           0.0000000
```

```
          U4       0.0000000              250.0000
          D1        15.00000             0.0000000
          D2       0.0000000              950.0000
          D3       0.0000000              1200.000
          D4       0.0000000              950.0000
          U5       0.0000000              1200.000
          D5        15.00000             0.0000000
          I0       0.0000000             0.0000000
          P0        40.00000             0.0000000
          P1        25.00000             0.0000000
          P2        25.00000             0.0000000
          P3        55.00000             0.0000000
          P4        55.00000             0.0000000
          P5        40.00000             0.0000000
         Row     Slack or Surplus      Dual Price
           1        43000.00             -1.000000
       CNDBI       0.0000000             -950.0000
       CNDBP       0.0000000             -600.0000
        INV1       0.0000000              950.0000
        INV2       0.0000000              250.0000
        INV3       0.0000000             -250.0000
        INV4       0.0000000             -950.0000
        CHG1       0.0000000              600.0000
        CHG2       0.0000000             -350.0000
        CHG3       0.0000000             -600.0000
        CHG4       0.0000000             -350.0000
        CHG5       0.0000000              600.0000
       CNDEI       0.0000000             -1650.000
       CNDEP       0.0000000              600.0000
```

We see the solution is a mixed policy:

$P_1 = P_2 = 25; \qquad P_3 = P_4 = 55.$

The mixed policy found by LP is $5,000 cheaper than the best pure policy.

9.2.3 Representing Absolute Values

You may be tempted to represent the production-change costs in the above model by the expression:

```
600 *( @ABS( P1 – P0) + @ABS( P2 – P1) + …+@ABS(P5 – P4));
```

This is mathematically correct, but computationally unwise, because it converts a linear program into a nonlinear program. Nonlinear programs are always more time consuming to solve. We have exploited the following result to obtain a linear program from an apparently nonlinear program. Subject to a certain condition, any appearance in a model of a term of the form:

@ABS (*expression*)

can be replaced by the term $U + D$, if we add the constraint:

$U – D = expression.$