SOS2 constraints in GLPK

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Additional note: piecewise linear functions expressed through SOS2 may be used to model not only non-linear *objectives*, but also non-linear equality and inequality *constraints*. This then allows general NLP (non-linear programming) problems to be reformulated as MIP (mixed-integer linear programming) problems.

SOS2 constraints: special ordered sets of type 2 (SOS2) constraints are normally used to model piecewise linear functions in convex and non-convex *separable programming*.

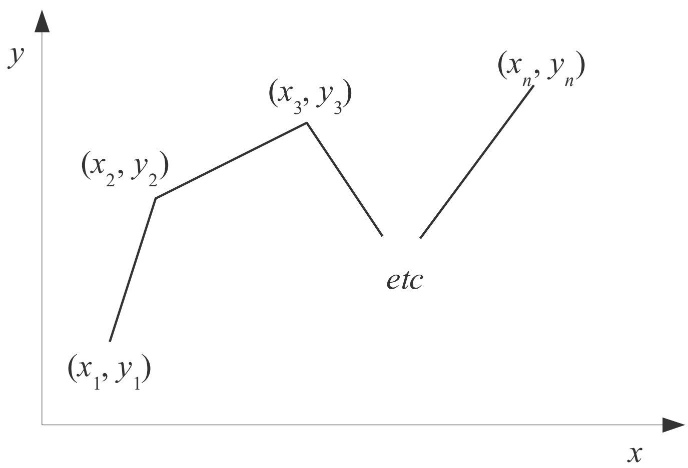
In the general case, an *SOS2 constraint* is completely defined by specifying a set of variables $\{ t_1, t_2, \ldots, t_n \}$ and this is equivalent to the following three constraints:

- $t_1, t_2, \ldots, t_n \geq 0$
- $t_1 + t_2 + \ldots + t_n = 1$
- only two adjacent variables, t_i and potentially t_{i+1}, can be non-zero.

Given that we need to model the *piecewise linear* continuous function

$$y = f(x)$$

specified by its n node points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ as shown below.



The standard description using an SOS2 constraint is the following:

- $x = x_1 t_1 + x_2 t_2 + \ldots + x_n t_n$
- $y = y_1 t_1 + y_2 t_2 + \ldots + y_n t_n$
- SOS2: $\{ t_1, t_2, \ldots, t_n \}$

where the SOS2 variables $t_1, t_2, \ldots, t_n$ play the role of interpolation parameters.

The implementation of SOS2 constraints within the simplex method assumes an additional rule to choose the variable to enter the basis. Namely, if t_i is basic, only t_{i-1} or t_{i+1} can be basic, while all other SOS2 variables have to be non-basic (and therefore fixed at zero).

However, since the set of feasible solutions may be non-convex, such a version of the simplex method enables only a local optimum to be obtained.

Modeling piecewise linear functions in GLPK: SOS2 constraints are not implemented in GLPK, but a piecewise linear function can be easily modeled using binary variables as follows.

Let $z_1, z_2, \ldots, z_{n-1}$ be binary variables, such that $z_i \in \{0, 1\}$, where:

- $z_i = 1$ means that $x_i \le x \le x_{i+1}$ and $y_i \le y \le y_{i+1}$

then, with $s_1, s_2, \ldots, s_{n-1}$ segment variables, such that $s_i \in \mathbb{R}$:

- $z_1 + z_2 + \ldots + z_{n-1} = 1$
- $0 \le s_i \le z_i$ for $i = 1, 2, \ldots, n-1$
- $$\begin{aligned} x = \; & + x_1 z_1 && + (x_2 - x_1)\, s_1 \\ & + x_2 z_2 && + (x_3 - x_2)\, s_2 \\ & \ldots \\ & + x_i z_i && + (x_{i+1} - x_i)\, s_i \\ & \ldots \\ & + x_{n-1} z_{n-1} && + (x_n - x_{n-1})\, s_{n-1} \end{aligned}$$
- $$\begin{aligned} y = \; & + y_1 z_1 && + (y_2 - y_1)\, s_1 \\ & + y_2 z_2 && + (y_3 - y_2)\, s_2 \\ & \ldots \\ & + y_i z_i && + (y_{i+1} - y_i)\, s_i \\ & \ldots \\ & + y_{n-1} z_{n-1} && + (y_n - y_{n-1})\, s_{n-1} \end{aligned}$$

The main advantage of this description is that the MIP solver is always able to find a global optimum.

Modeling SOS2 constraints in GLPK: if necessary, SOS2 constraints can be modeled independently when modeling a piecewise linear function thus,

Let $\{ t_1, t_2, \ldots, t_n \}$ be an SOS2 constraint. Then its equivalent description is the following:

- $z_1 + z_2 + \ldots + z_{n-1} = 1$
- $0 \le s_i \le z_i$ for $i = 1, 2, \ldots, n-1$
- $$\left[\begin{aligned} t_1 \; & = z_1 - s_1 \\ t_2 \; & = z_2 - s_2 + s_1 \\ \ldots & \\ t_i \; & = z_i - s_i + s_{i-1} \\ \ldots & \\ t_{n-1} \; & = z_{n-1} - s_{n-1} + s_{n-2} \\ t_n \; & = s_{n-1} \end{aligned} \right.$$

where $z_1, z_2, \ldots, z_{n-1}$ are binary variables and $z_i = 1$ means that only t_i and potentially t_{i+1} are non-zero. ❑

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