

# Calculation of the Zernike polynomials and their partial derivatives in Cartesian coordinates using the standard indexes OSA/ANSI.

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July 21th, 2019

## 1 Introduction

The `zernikes_and_derivatives_cartesian_OSA` function calculates the Zernike polynomials and their partial derivatives in Cartesian coordinates. For this purpose, it uses the algorithm described in reference [1]. However, it should be noted that in the cited article the author use unit-normalized Zernike polynomials arranged according to the azimuthal scheme set forth by Rimmer and Wyant [2], while to implement the function has been used the OSA/ANSI standard notation described in references [3, 4]. This means the index scheme is different and, moreover, the polynomials are not unit-normalized but normalized to  $\pi$ .

## 2 Notation

Zernike's polynomials are usually ordered by a double-index,  $Z_n^m$ , being  $n$  the radial order and  $m$  the angular frequency, both integers. Although in programming it is usually utilized a single index,  $Z_j$ .

The scheme used in reference [1], consider a double-index with  $n \geq 0$  and  $0 \leq m \leq n$ . However, the standard OSA/ANSI<sup>1</sup> uses a double-index with  $n \geq 0$  and  $-n \leq m \leq n$ , as shown in the following table:

Rimmer&Wyant schema		OSA standard	
$\tilde{n}$	$\tilde{m}$	$n$	$m$
0	0	0	0
1	0, 1	1	-1, 1
2	0, 1, 2	2	-2, 0, 2
3	0, 1, 2, 3	3	-3, -1, 1, 3
4	0, 1, 2, 3, 4	4	-4, -2, 0, 2, 4
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Therefore, by the way of example, the following polynomials –using the two schemes seen– are equivalent:

Rimmer&Wyant schema	OSA standard
$Z_1^0(x, y)$	$Z_1^{-1}(x, y)$
$Z_3^0(x, y)$	$Z_3^{-3}(x, y)$
$Z_3^1(x, y)$	$Z_3^{-1}(x, y)$
$Z_4^4(x, y)$	$Z_4^4(x, y)$
$Z_8^6(x, y)$	$Z_8^4(x, y)$

Using  $(\tilde{n}, \tilde{m})$  to denote the R&W indices, and  $(n, m)$  those of the OSA scheme, both pairs of indices are related as follow:

$$n = \tilde{n} \quad m = 2 \cdot \tilde{m} - \tilde{n} \quad (1)$$

<sup>1</sup>In fact, this double index schema is the most usual, and is employed by other notations than the standard OSA/ANSI.

## 2.1 Single index

As said before, it is more convenient to use a simple index to program. In the OSA standard, the conversion of indices is as follows:

$$j = \frac{n(n+2) + m}{2} \quad n = \text{roundup} \left[ \frac{-3 + \sqrt{9 + 8j}}{2} \right]$$

$$m = 2j - n(n+2)$$

## 2.2 U-polynomials

Employing the notation used in reference [1], the unit-normalized Zernike's polynomials are called  $U_{nm}$ , and are related to the  $\pi$ -normalized ones as follows:

$$Z_n^m(x, y) = N_{nm} \cdot U_{nm}(x, y)$$

with

$$N_{nm} = \sqrt{\frac{2(n+1)}{1 + \delta_{m0}}} \quad \text{being} \quad \delta_{m0} = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases}$$

## 3 Algorithm

It is well known that for high-order Zernike's polynomials, computing the values using explicit expressions is not the best strategy, since it is inefficient and suffer from large cancellation errors. So, several schemes using recurrence relations have been devised.

Reference [1] presents one recurrence relation with coefficients that do not depend on radial or azimuthal orders and which contains no singularities. In addition, it also presents a recurrence relation to compute the partial derivatives in Cartesian coordinates.

### 3.1 Recurrence relations in Cartesian coordinates for Zernike U-polynomials

The general recurrence relation is

$$U_{\tilde{n}, \tilde{m}} = xU_{\tilde{n}-1, \tilde{m}} + yU_{\tilde{n}-1, \tilde{n}-1-\tilde{m}} + xU_{\tilde{n}-1, \tilde{m}-1} - yU_{\tilde{n}-1, \tilde{n}-\tilde{m}} - U_{\tilde{n}-2, \tilde{m}-1}$$

But there are several exceptions:

- for  $\tilde{m} = 0$       $U_{\tilde{n}, 0} = xU_{\tilde{n}-1, 0} + yU_{\tilde{n}-1, \tilde{n}-1}$
- for  $\tilde{m} = \tilde{n}$       $U_{\tilde{n}, \tilde{n}} = xU_{\tilde{n}-1, \tilde{n}-1} - yU_{\tilde{n}-1, 0}$
- for  $\tilde{n}$  odd and  $\tilde{m} = \frac{\tilde{n}-1}{2}$

$$U_{\tilde{n}, \tilde{m}} = yU_{\tilde{n}-1, \tilde{n}-1-\tilde{m}} + xU_{\tilde{n}-1, \tilde{m}-1} - yU_{\tilde{n}-1, \tilde{n}-\tilde{m}} - U_{\tilde{n}-2, \tilde{m}-1}$$

- for  $\tilde{n}$  odd and  $\tilde{m} = \frac{\tilde{n}-1}{2} + 1$

$$U_{\tilde{n}, \tilde{m}} = xU_{\tilde{n}-1, \tilde{m}} + yU_{\tilde{n}-1, \tilde{n}-1-\tilde{m}} + yU_{\tilde{n}-1, \tilde{m}-1} - U_{\tilde{n}-2, \tilde{m}-1}$$

- for  $\tilde{n}$  even and  $\tilde{m} = \frac{\tilde{n}}{2}$

$$U_{\tilde{n}, \tilde{m}} = 2xU_{\tilde{n}-1, \tilde{m}} + 2yU_{\tilde{n}-1, \tilde{m}-1} - U_{\tilde{n}-2, \tilde{m}-1}$$

The starting polynomials being

$$U_{0,0} = 1, \quad U_{1,0} = y, \quad U_{1,1} = x \quad .$$

### 3.2 Recurrence relations for the OSA/ANSI scheme

If we now “translate” the previous equations into the OSA scheme, we obtain that the general recurrence relation is

$$U_{n,m} = xU_{n-1,m+1} + yU_{n-1,-(m+1)} + xU_{n-1,m-1} - yU_{n-1,1-m} - U_{n-2,m}$$

the exceptions are given by

- for  $m = -n$      $U_{n,-n} = xU_{n-1,1-n} + yU_{n-1,n-1}$
- for  $m = n$      $U_{n,n} = xU_{n-1,n-1} - yU_{n-1,1-n}$
- for  $m = -1$      $U_{n,-1} = yU_{n-1,0} + xU_{n-1,-2} - yU_{n-1,2} - U_{n-2,-1}$
- for  $m = 1$      $U_{n,1} = xU_{n-1,2} + yU_{n-1,-2} + xU_{n-1,0} - U_{n-2,1}$
- for  $m = 0$      $U_{n,0} = 2xU_{n-1,1} + 2yU_{n-1,-1} - U_{n-2,0}$

and the starting polynomials are

$$U_{0,0} = 1, \quad U_{1,-1} = y \quad U_{1,1} = x \quad .$$

Note that using the OSA scheme the conditions of the exceptions have been simplified, which makes it easier to compute.

#### 3.2.1 Recurrence relations with a single index

As mentioned earlier, when programming it is more convenient to use a simple index. In this case, recurrence relations are given as follows. Let us start now with the exceptions:

- for  $m = -n \Rightarrow j = \frac{(n+1)n}{n}$      $U_j = xU_{j-n} + yU_{j-1}$
- for  $m = n \Rightarrow j = \frac{n(n+3)}{2}$      $U_j = xU_{j-(n+1)} - yU_{j-2n}$
- for  $m = -1 \Rightarrow j = \frac{n(n+2)-1}{2}$      $U_j = xU_{j-(n+1)} + yU_{j-n} - yU_{j-(n-1)} - U_{j-2n}$
- for  $m = 1 \Rightarrow j = \frac{n(n+2)+1}{2}$      $U_j = xU_{j-n} + xU_{j-(n+1)} + yU_{j-(n+2)} - U_{j-2n}$
- for  $m = 0 \Rightarrow j = \frac{n(n+2)}{2}$      $U_j = 2xU_{j-n} + 2yU_{j-(n+1)} - U_{j-2n} .$

The general case is

$$U_j = xU_{j-n} + yU_{j-(n+m+1)} + xU_{j-(n+1)} - yU_{j-(m+m)} - U_{j-2n}$$

and the starting polynomials are

$$U_0 = 1, \quad U_1 = y, \quad U_2 = x \quad .$$

### 3.3 Recurrence relations for Cartesian derivatives for Zernike U-polynomials

The general recursive relations for partial derivatives are:

$$\frac{\partial U_{\tilde{n},\tilde{m}}}{\partial x} = \tilde{n}U_{\tilde{n}-1,\tilde{m}} + \tilde{n}U_{\tilde{n}-1,\tilde{m}-1} + \frac{\partial U_{\tilde{n}-2,\tilde{m}-1}}{\partial x}$$

and

$$\frac{\partial U_{\tilde{n},\tilde{m}}}{\partial y} = \tilde{n}U_{\tilde{n}-1} - \tilde{n}U_{\tilde{n}-1,\tilde{n}-\tilde{m}} + \frac{\partial U_{\tilde{n}-2,\tilde{m}-1}}{\partial y}$$

But, as before, there are several exceptions:

- for  $\tilde{m} = 0$

$$\frac{\partial U_{\tilde{n},0}}{\partial x} = \tilde{n}U_{\tilde{n}-1,0} \quad \frac{\partial U_{\tilde{n},0}}{\partial y} = \tilde{n}U_{\tilde{n}-1,\tilde{n}-1}$$

- for  $\tilde{m} = \tilde{n}$

$$\frac{\partial U_{\tilde{n},\tilde{n}}}{\partial x} = -\tilde{n}U_{\tilde{n}-1,\tilde{n}-1} \quad \frac{\partial U_{\tilde{n},\tilde{n}}}{\partial y} = -\tilde{n}U_{\tilde{n}-1,0}$$

- for  $\tilde{n}$  odd and  $\tilde{m} = \frac{\tilde{n}-1}{2}$

$$\frac{\partial U_{\tilde{n},\tilde{m}}}{\partial x} = \tilde{n}U_{\tilde{n}-1,\tilde{m}-1} + \frac{\partial U_{\tilde{n}-2,\tilde{m}-1}}{\partial x} \quad \frac{\partial U_{\tilde{n},\tilde{m}}}{\partial y} = \tilde{n}U_{\tilde{n}-1,\tilde{n}-\tilde{m}-1} - \tilde{n}U_{\tilde{n}-1,\tilde{n}-\tilde{m}} + \frac{\partial U_{\tilde{n}-2,\tilde{m}-1}}{\partial y}$$

- for  $\tilde{n}$  odd and  $\tilde{m} = \frac{\tilde{n}-1}{2} + 1$

$$\frac{\partial U_{\tilde{n},\tilde{m}}}{\partial x} = \tilde{n}U_{\tilde{n}-1,\tilde{m}} + \tilde{n}U_{\tilde{n}-1,\tilde{m}-1} + \frac{\partial U_{\tilde{n}-2,\tilde{m}-1}}{\partial x} \quad \frac{\partial U_{\tilde{n},\tilde{m}}}{\partial y} = \tilde{n}U_{\tilde{n}-1,\tilde{n}-\tilde{m}-1} + \frac{\partial U_{\tilde{n}-2,\tilde{m}-1}}{\partial y}$$

- for  $\tilde{n}$  even and  $\tilde{m} = \frac{\tilde{n}}{2}$

$$\frac{\partial U_{\tilde{n},\tilde{m}}}{\partial x} = 2\tilde{n}U_{\tilde{n}-1,\tilde{m}} + \frac{\partial U_{\tilde{n}-2,\tilde{m}-1}}{\partial x} \quad \frac{\partial U_{\tilde{n},\tilde{m}}}{\partial y} = 2\tilde{n}U_{\tilde{n}-1,\tilde{n}-\tilde{m}-1} + \frac{\partial U_{\tilde{n}-2,\tilde{m}-1}}{\partial y}$$

The starting expressions for the Cartesian derivatives being:

$$\frac{\partial U_{0,0}}{\partial x} = \frac{\partial U_{0,0}}{\partial y} = 0, \quad \frac{\partial U_{1,0}}{\partial x} = \frac{\partial U_{1,1}}{\partial y} = 0, \quad \frac{\partial U_{1,1}}{\partial x} = \frac{\partial U_{1,0}}{\partial y} = 1$$

#### 3.3.1 Recurrence relations for the OSA/ANSI scheme

If we now “translate” the previous equations into the OSA scheme, we obtain that the general recurrence relations for partial derivatives are

$$\frac{\partial U_{n,m}}{\partial x} = nU_{n-1,m+1} + nU_{n-1,m-1} + \frac{\partial U_{n-2,m}}{\partial x}$$

and

$$\frac{\partial U_{n,m}}{\partial y} = nU_{n-1,-(m+1)} - nU_{n-1,1-m} + \frac{\partial U_{n-2,m}}{\partial y}$$

the exceptions are give by

- for  $m = -n$   $\frac{\partial U_{n,-n}}{\partial x} = nU_{n-1,1-n}$   $\frac{\partial U_{n,-n}}{\partial y} = nU_{n-1,n-1}$

- for  $m = n$   $\frac{\partial U_{n,-n}}{\partial x} = nU_{n-1,n-1}$   $\frac{\partial U_{n,-n}}{\partial y} = -nU_{n-1,1-n}$

- for  $m = -1$

$$\frac{\partial U_{n,-1}}{\partial x} = nU_{n-1,-2} + \frac{\partial U_{n-2,-1}}{\partial x} \quad \frac{\partial U_{n,-1}}{\partial y} = nU_{n-1,0} - nU_{n-1,2} + \frac{\partial U_{n-2,-1}}{\partial y}$$

- for  $m = 1$

$$\frac{\partial U_{n,1}}{\partial x} = nU_{n-1,2} + nU_{n-1,0} + \frac{\partial U_{n-2,1}}{\partial x} \quad \frac{\partial U_{n,1}}{\partial y} = nU_{n-1,-2} + \frac{\partial U_{n-2,1}}{\partial y}$$

- for  $m = 0$

$$\frac{\partial U_{n,0}}{\partial x} = 2nU_{n-1,1} + \frac{\partial U_{n-2,0}}{\partial x} \quad \frac{\partial U_{n,0}}{\partial y} = 2nU_{n-1,-1} + \frac{\partial U_{n-2,0}}{\partial y}$$

and the starting derivatives are

$$\frac{\partial U_{0,0}}{\partial x} = \frac{\partial U_{0,0}}{\partial y} = 0, \quad \frac{\partial U_{1,-1}}{\partial x} = \frac{\partial U_{1,1}}{\partial y} = 0, \quad \frac{\partial U_{1,1}}{\partial x} = \frac{\partial U_{1,-1}}{\partial y} = 1$$

### 3.3.2 Recurrence relations with a single index

Let us start now with the exceptions:

- for  $m = -n \Rightarrow j = \frac{(n+1)n}{n}$

$$\frac{\partial U_j}{\partial x} = nU_{j-n} \quad \frac{\partial U_j}{\partial y} = nU_{j-1}$$

- for  $m = n \Rightarrow j = \frac{n(n+3)}{2}$

$$\frac{\partial U_j}{\partial x} = nU_{j-(n+1)} \quad \frac{\partial U_j}{\partial y} = -nU_{j-2n}$$

- for  $m = -1 \Rightarrow j = \frac{n(n+2)-1}{2}$

$$\frac{\partial U_j}{\partial x} = nU_{j-(n+1)} + \frac{\partial U_{j-2n}}{\partial x} \quad \frac{\partial U_j}{\partial y} = nU_{j-n} - nU_{j-(n-1)} + \frac{\partial U_{j-2n}}{\partial y}$$

- for  $m = 1 \Rightarrow j = \frac{n(n+2)+1}{2}$

$$\frac{\partial U_j}{\partial x} = nU_{j-n} + nU_{j-(n+1)} + \frac{\partial U_{j-2n}}{\partial x} \quad \frac{\partial U_j}{\partial y} = nU_{j-(n+2)} + \frac{\partial U_{j-2n}}{\partial y}$$

- for  $m = 0 \Rightarrow j = \frac{n(n+2)}{2}$

$$\frac{\partial U_j}{\partial x} = 2nU_{j-n} + \frac{\partial U_{j-2n}}{\partial x} \quad \frac{\partial U_j}{\partial y} = 2nU_{j-(n+1)} + \frac{\partial U_{j-2n}}{\partial y}$$

The general case is

$$\frac{\partial U_j}{\partial x} = nU_{j-n} + nU_{j-(n+1)} + \frac{\partial U_{j-2n}}{\partial x} \quad \frac{\partial U_j}{\partial y} = nU_{j-(n+m+1)} - nU_{j-(n+m)} + \frac{\partial U_{j-2n}}{\partial y}$$

and the starting derivatives are

$$\frac{\partial U_0}{\partial x} = \frac{\partial U_0}{\partial y} = 0, \quad \frac{\partial U_1}{\partial x} = \frac{\partial U_2}{\partial y} = 0, \quad \frac{\partial U_2}{\partial x} = \frac{\partial U_1}{\partial y} = 1$$

## References

- [1] Andersen T.B., [Efficient and robust recurrence relations for the Zernike circle polynomials and their derivatives in Cartesian coordinates](#). *Optic Express* 26(15), 18878-18896 (2018).
- [2] Rimmer M.P. & Wyant J.C., [Evaluation of large aberrations using a lateral-shear interferometer having variable shear](#). *Appl. Opt.* 14(1), 142-150 (1975).
- [3] Thibos, L.N, Applegate, R.A., Schwiegerling, J.T. & Webb, R., [Standards for reporting the optical aberrations of eyes](#). *Journal of refractive surgery*, 18(5), S652-S660 (2002).
- [4] ANSI Z80.28-2017. American National Standards of Ophthalmics: methods for reporting optical aberrations of eyes.