Calculus III — MTH 253 November 24, 2005

Midterm 2

Name : _____ **Score :** (/140) = _____%

Problem 1. Find the equation for a sphere that passes through the point $(4, 3, -1)$ and has its center at $(3, 8, 1)$.

(/10)

$$\begin{aligned} \boldsymbol{\rho} &= \langle 4-3, 3-8, -1-1 \rangle \\ &= \langle 1, -5, -2 \rangle \\ \|\boldsymbol{\rho}\| &= \sqrt{(1)^2 + (-5)^2 + (-2)^2} \\ &= \sqrt{30} \\ (x-3)^2 + (y-8)^2 + (z-1)^2 &= 30 \end{aligned}$$

Problem 2. If $\mathbf{a} = 2\,\mathbf{i} - 3\,\mathbf{j}$, and $\mathbf{b} = \mathbf{i} + 5\,\mathbf{j}$,

a) Find $|\mathbf{a}|$

(/5)

$$\begin{aligned} \mathbf{a} &= \langle 2, -3 \rangle \\ |\mathbf{a}| &= \sqrt{(2)^2 + (-3)^2} \\ &= \sqrt{13} \end{aligned}$$

b) Find $3\,\mathbf{a} + 4\,\mathbf{b}$

(/5)

$$\begin{aligned} \mathbf{a} &= \langle 2, -3 \rangle \\ 3\,\mathbf{a} &= 3\,\langle 2, -3 \rangle \\ &= \langle 6, -9 \rangle \\ \mathbf{b} &= \langle 1, 5 \rangle \\ 4\,\mathbf{b} &= 4\,\langle 1, 5 \rangle \\ &= \langle 4, 20 \rangle \\ 3\,\mathbf{a} + 4\,\mathbf{b} &= \langle 6, -9 \rangle + \langle 4, 20 \rangle \\ &= \langle 6 + 4, 20 - 9 \rangle \\ &= \langle 10, 11 \rangle \\ &= 10\,\mathbf{i} + 11\,\mathbf{j} \end{aligned}$$

Problem 3. For $\mathbf{a} = \mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}$, $\mathbf{b} = 5\,\mathbf{i} + 9\,\mathbf{k}$,

a) Find $\mathbf{a} \cdot \mathbf{b}$

(/5)

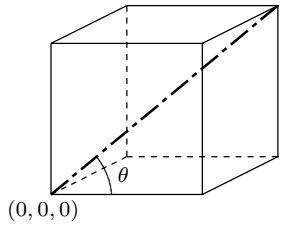
$$\begin{aligned} \mathbf{a} &= \langle 1, -2, 3 \rangle \\ \mathbf{b} &= \langle 5, 0, 9 \rangle \\ \mathbf{a} \cdot \mathbf{b} &= 1\,(5) + (-2)\,0 + 3\,(9) \\ &= 32 \end{aligned}$$

b) Find the scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$.

(/5)

$$\begin{aligned} \text{proj}_{\mathbf{a}}\,\mathbf{b} &= \frac{|\mathbf{a} \cdot \mathbf{b}|}{\|\mathbf{a}\|^2}\,\mathbf{a} \\ \|\mathbf{a}\| &= \sqrt{(1)^2 + (-2)^2 + (3)^2} \\ \|\mathbf{a}\|^2 &= 14 \\ \frac{|\mathbf{a} \cdot \mathbf{b}|}{\|\mathbf{a}\|^2} &= \frac{32}{14} = \frac{16}{7} \qquad \text{Scalar projection of } \mathbf{b} \text{ onto } \mathbf{a}. \\ \text{proj}_{\mathbf{a}}\,\mathbf{b} &= \frac{16}{7}\,\langle 1, -2, 3 \rangle \\ &= \left\langle \frac{16}{7}, -\frac{32}{7}, \frac{48}{7} \right\rangle \qquad \text{Vector projection of } \mathbf{b} \text{ onto } \mathbf{a}. \end{aligned}$$

Problem 4. Find the angle between the main diagonal of a unit-cube and one of it's edges, to the nearest degree.



(/10)

Let $O = (0, 0, 0)$ be the origin, $P = (0, 1, 0)$ be the lower right corner of the side facing the viewer, and $Q = (-1, 1, 1)$ be the upper right corner of the opposite side, where the bold dash-dotted line representing the diagonal of the cube ends. $\triangle OPQ$ is a right triangle, and thus ordinary trigonometric ratios apply to it.

$$\begin{aligned}
\overrightarrow{OP} &= \langle 0, 1, 0 \rangle \\
\|\overrightarrow{OP}\| &= 1 \\
\overrightarrow{OQ} &= \langle -1, 1, 1 \rangle \\
\|\overrightarrow{OQ}\| &= \sqrt{(-1)^2 + (1)^2 + (1)^2} \\
&= \sqrt{3} \\
\|\overrightarrow{OP}\| &= \|\overrightarrow{OQ}\| \cos\theta \\
1 &= \sqrt{3} \cos\theta \\
\theta &= \cos^{-1} \frac{1}{\sqrt{3}} \\
&\approx 0.95532 \,\text{rad} \\
&\approx 54.7356^\circ \\
&\approx 55^\circ
\end{aligned}$$

Problem 5. Given the points $P(1, 0, 0)$, $Q(0, 2, 0)$, and $R(0, 0, 3)$:

a) Find a vector orthogonal to the plane containing P, Q, and R.

(/10)

The cross product always produces a vector that is orthogonal to both input vectors.

$$\begin{aligned}
\overrightarrow{PQ} &= \langle -1, 2, 0 \rangle \\
\overrightarrow{PR} &= \langle -1, 0, 3 \rangle
\end{aligned}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} \;=\; \langle 6, 3, 2 \rangle$$

```
→ cross(vector([-1,2,0]),vector([-1,0,3]))
```

[6, 3, 2] (6)

Type: Vector Integer

→

b) What is the area of triangle $\triangle PQR$?

(/10)

The area of the triangle formed by two vectors is half the area of the parallelogram formed by them. The area of a parallelogram formed by two vectors is equal to their dot product.

$$\begin{aligned} \langle -1, 2, 0 \rangle \cdot \langle -1, 0, 3 \rangle &= 1 \\ \therefore \quad \text{Area of } \triangle PQR &= \frac{1}{2} \text{units}^2 \end{aligned}$$

```
→ dot(vector([-1,2,0]),vector([-1,0,3]))
```

1 (1)

Type: PositiveInteger

→

Problem 6. Write a parametric and symmetric equations of a line passing through the points $(3, 1, -1)$ and $(3, 2, -6)$.

(/20)

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_0 + t\,\mathbf{v} \\ \text{Let } \mathbf{r} &= \langle x.y, z \rangle \\ \text{Let } \mathbf{r}_0 &= \langle 3, 1, -1 \rangle \\ \text{Let } \mathbf{r}_1 &= \langle 3, 2, -6 \rangle \\ \text{Let } \mathbf{v} &= \mathbf{r}_1 - \mathbf{r}_0 = \langle 3-3, 2-1, -6-(-1) \rangle \\ &= \langle 0, 1, -5 \rangle \\ \langle x, y, z \rangle &= \langle 3, 1, -1 \rangle + t\,\langle 0, 1, -5 \rangle \\ x &= 3 \\ y &= 1 + t \\ z &= -1 - 5\,t \\ x = 3 \quad \frac{y-1}{1} &= \frac{z+1}{-5} \end{aligned}$$

Problem 7. Write the equation of the plane which contains the point $(6,3,2)$ and is perpendicular to the vector $\langle -2,1,5\rangle$.

(/10)

$$\begin{aligned} -2\,(x-6)+(y-3)+5\,(z-2) &= 0 \\ -2\,x+y+5\,z-1 &= 0 \end{aligned}$$

Problem 8. Find the point at which the line $x=1+2\,t$, $y=-1$, $z=t$ intersects the plane $2\,x-y+z-5=0$.

(/10)

$$\begin{aligned} 2\,(1+2\,t)-(-1)+(t)-5 &= 0 \\ 5\,t-2 &= 0 \\ t &= \frac{2}{5} \\ x &= 1+2\left(\frac{2}{5}\right) \\ y &= -1 \\ z &= \frac{2}{5} \end{aligned}$$

$$\therefore \quad \text{Point of intersection is} \quad \left(\frac{9}{5},-1,\frac{2}{5}\right)$$

Problem 9. State the *binomial theorem*, and then use it to expand $f(x)=\sqrt[3]{1+x^2}$ as a power series, writing the individual terms up to degree 3.

(/10)

The Binomial Series states that if k is any real number and $|x|<1$, then

$$\begin{aligned} (1+x)^k &= 1+k\,x+\frac{k\,(k-1)}{2!}\,x^2+\frac{k\,(k-1)\,(k-2)}{3!}\,x^3+\ldots. \\ &= \sum_{n=0}^{\infty}\binom{k}{n}\,x^n \end{aligned}$$

$$\text{where } \binom{k}{n} = \frac{k\,(k-1)\cdots\big(k-(n-1)\big)}{n!} \qquad (n\geqslant 1) \quad \text{and} \quad \binom{k}{0}=1$$

$$\begin{aligned} \sqrt[3]{1+x^2} &= \left(1+x^2\right)^{\frac{1}{3}} \\ &= \sum_{n=0}^{\infty}\binom{\frac{1}{3}}{n}\left(x^2\right)^n \\ &= \sum_{n=0}^{\infty}\binom{\frac{1}{3}}{n}\,x^{2n} \\ &= 1+\frac{\frac{1}{3}}{1!}\,x^{2(1)} \\ &\quad +\frac{\frac{1}{3}\left(\frac{1}{3}-(2-1)\right)}{2!}\,x^{2(2)} \\ &\quad +\frac{\frac{1}{3}\left(\frac{1}{3}-(3-2)\right)\left(\frac{1}{3}-(3-1)\right)}{3!}\,x^{2(3)} \\ &\quad +\frac{\frac{1}{3}\left(\frac{1}{3}-(4-3)\right)\left(\frac{1}{3}-(4-2)\right)\left(\frac{1}{3}-(4-1)\right)}{4!}\,x^{2(4)}+\cdots \\ &= 1+\frac{1}{3}x^2-\frac{1}{9}x^4+\cdots \end{aligned}$$

Problem 10. Find the fourth degree *Taylor approximation*, $T_4(x)$, for the function $f(x)=\cos x$ centered at the point $a=\frac{\pi}{3}$. What is the maximum error, $R_4(x)$, on the interval $\left[0,\frac{2\pi}{3}\right]$?

(/10)

$$T_n(x)\Big|_a = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$\begin{aligned}
f^{(0)}(x) &= \cos x & f^{(0)}(a) &= \frac{1}{2}\\
f^{(1)}(x) &= -\sin x & f^{(1)}(a) &= -\frac{\sqrt{3}}{2}\\
f^{(2)}(x) &= -\cos x & f^{(2)}(a) &= -\frac{1}{2}\\
f^{(3)}(x) &= \sin x & f^{(3)}(a) &= \frac{\sqrt{3}}{2}\\
f^{(4)}(x) &= \cos x & f^{(4)}(a) &= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\cos x &= \frac{\frac{1}{2}}{0!}\left(x-\frac{\pi}{3}\right)^0 + \frac{-\frac{\sqrt{3}}{2}}{1!}\left(x-\frac{\pi}{3}\right)^1 + \frac{-\frac{1}{2}}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{\frac{\sqrt{3}}{2}}{3!}\left(x-\frac{\pi}{3}\right)^3 + \frac{\frac{1}{2}}{4!}\left(x-\frac{\pi}{3}\right)^4 + \cdots\\
&\approx \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right) - \frac{1}{4}\left(x-\frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x-\frac{\pi}{3}\right)^3 + \frac{1}{48}\left(x-\frac{\pi}{3}\right)^4
\end{aligned}$$

Taylor's inequality says that if $|f^{(n+1)}(x)| \leqslant M$ for $|x-a| \leqslant d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality:

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leqslant d$$

The method of solution then is to find $f^{(n+2)}(x)$, and then find it's roots inside the domain segment under consideration. That tells you the maximums, minimums, and inflection points of $f^{(n+1)}(x)$. To that list, add the end points, and then for each item on this list, evaluate $\left|f^{(n+1)}(x)\right|$, and find the maximum result. Assign that value to M, and then evaluate the expression on the right hand side of Taylor's inequality with $x=0$ to find the maximum error.

```
→ D(cos(x), x, 4 + 2)
```

$$-\cos(x) \tag{1}$$

Type: Expression Integer

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→ zeros := solve(-cos(x) = 0, x)
```

$$\left[x=\frac{\pi}{2}\right] \tag{2}$$

Type: List Equation Expression Integer

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→ try : List Float := append([0,((2*%pi)/3)],[(rhs(E)) for E in zeros])
```

$$[0.0,\ 2.0943951023931954923,\ 1.5707963267948966192] \tag{3}$$

Type: List Float

```
→ D(cos(x),x,4+1)
```

$$-\sin(x) \tag{4}$$

Type: Expression Integer

```
→ M := reduce(max,[(abs(-sin(x))) for x in try])
```

1.0 (5)

Type: Float

```
→ eval( (M/factorial(4 + 1))*abs(x - (%pi/3))^(4 + 1), x=0)
```

0.010494502221717470962 (6)

Type: Expression Float

→

The maximum error is expression (6), above.